2 Principle of Mathematical Induction

Let we want to show that property P holds for all natural numbers. To prove this property, P using mathematical induction, following are the steps:
Basic Step:
First show that property P is true for 0 or 1.
Induction Hypothesis:
Assume that property P holds for n.
Induction step:
Using induction hypothesis, show that P is true for n+1.
Then by the principle of mathematical induction, P is true for all natural numbers.

Examples

Example 1
Prove using mathematical induction, $n^4 - 4n^2$ is divisible by 3 for n>=0.
Basic step:
For n=0,
$n^4 - 4n^2 = 0$, which is divisible by 3.
Induction hypothesis:
Let $n^4 - 4n^2$ is divisible by 3.
Induction step:

$$(n+1)^4 - 4(n+1)^2 = [(n+1)^2]^2 - 4(n+1)^2$$
$$= (n^2 + 2n + 1)^2 - (2n + 2)^2$$
$$= (n^2 + 2n + 1 + 2n + 2)(n^2 + 2n + 1 - 2n - 2)$$
$$= (n^2 + 4n + 3)(n^2 - 1)$$
$$= n^4 + 4n^3 + 3n^2 - 3 - 4n - n^2 = n^4 + 4n^3 + 2n^2 - 4n - 3$$
$$= n^4 + 4n^3 - 4n^2 + 6n^2 - 4n - 3$$
$$= n^4 - 4n^2 + 6n^2 - 3 + 4n^3 - 4n$$
$$= (n^4 - 4n^2) + (6n^2) - (3) + 4(n^3 - n)$$

$(n^4 - 4n^2)$ is divisible by 3 from our hypothesis.
$6n^2$ and 3 are divisible by 3.
We need to prove that $4(n^3 - n)$ is divisible by 3.

Again use mathematical induction.

Basic step:
For n = 0,
4(0-0) = 0 is divisible by 3.
Induction hypothesis:
Let $4(n^3 - n)$ is divisible by 3.
Induction step:

$$4[(n+1)^3 - (n+1)] = 4[(n^3 + 3n^2 + 3n + 1) - (n + 1)]$$
$$= 4[n^3 + 3n^2 + 3n + 1 - n - 1]$$

$= 4[n^3 + 3n^2 + 2n]$
$= 4[n^3 - n + 3n^2 + 3n]$
$= 4(n^3 - n) + 4.3n^2 + 4.3n$

$4(n^3 - n)$ is divisible by 3 from our hypothesis.
$4.3n^2$ is divisible by 3.
4.3n is divisible by 3.
Thus we can say that
$(n^2 - 4n^2) + (6n^2) - (3) + 4(n^3 - n)$ is divisible by 3.
That is,
$n^4 - 4n^2$ is divisible by 3.

Example 2:
Prove using mathematical induction:
$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
Let $P\{n) = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
Basic Step:
For n=1,
LHS =1
RHS = $\frac{1(1+1)}{2}$=1 Induction hypothesis;
Assume that P(n) is true for n=k,
Then,
$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

Induction step:
Show that P(n) is true for n=k+1.
$1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + k + 1$
$= (k + 1)\left(\frac{k}{2} + 1\right)$
$=(k + 1)\left[\frac{k+2}{2}\right]$

$= \frac{[(k+1)(k+2)]}{2}$

That is, P(n) is true for n=k+1.
Thus by using the principle of mathematical induction, we proved $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Example 3:
Prove using the principle of mathematical induction,
$\sum_{i=0}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$
Let $P(n) = \sum_{i=0}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$

Basic step:
For n=1,

LHS $1^2=1$

RHS = $\frac{n(n+1)(2n+1)}{6}$ = $\frac{1.2.3}{6}$ =1

P(n) is true for n=1.

Induction hypothesis:

Assume that result is true for n=k.

That is,

$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Induction Step:

Prove that result is true for n=k+1.

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= (k+1)\left[\left(\frac{k(2k+1)}{6}\right) + (k+1)\right]$$
$$= \frac{(k+1)(2k^2+k+6k+6)}{6}$$
$$= \frac{(k+1)(2k^2+7k+6)}{6}$$
$$= \frac{(k+1)(2k(k+2)+3(k+2))}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Thus it is proved.

Exercise Problems:

1. Prove the following by principle of induction:

- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
- 2i>I for all i>1.
- $2^x \geq x^2$, if $x \geq 4$
- 10^{2n} -1 is divisible by 11 for all n>1
- $2^n > n$, for all n>1